Name: ____________________

Math 10550, Final Exam: Solutions
December 18, 2016

Instructor: ____________________

- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.
- No calculators are to be used.
- The exam lasts for two hours.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 22 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)	15.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)	16.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)	17.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)	18.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)	19.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)	21.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)	22.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)	23.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)	24.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)	25.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)						
13.	(a)	(b)	(c)	(d)	(e)						
14.	(a)	(b)	(c)	(d)	(e)						

Name: ______________________________

Instructor: __________________________

Multiple Choice

1.(6 pts.) Find the limit

$$\lim_{x \to 3^-} \frac{x^2 - 4x + 3}{x - 3}.$$

(a) 1 (b) ∞ (c) $-\infty$ (d) 2 (e) -1

2.(6 pts.) Find the limit

$$\lim_{x \to 2^+} \frac{\sqrt{x^2 - 1} - \sqrt{x + 1}}{x - 2}.$$

(a) $-\infty$ (b) ∞ (c) $\dfrac{1}{2\sqrt{3}}$ (d) 3 (e) $\dfrac{\sqrt{3}}{2}$

Name: ______________________________

Instructor: __________________________

3.(6 pts.) Find the equation of the tangent line to the curve $y = \sqrt{2x+1}$ at $x = 4$.

Solution: To compute the equation for the tangent line to the curve, we need a slope and a point on the line. To get the slope of the tangent line at the point with x coordinate 4, we evaluate the derivative at 4. The derivative is given by $y' = \frac{1}{\sqrt{2x+1}}$. We see that $y'(4) = 1/3$. We also have that $y(4) = 3$. Thus, the point-slope form of the tangent line is given by $y - 3 = (1/3)(x-4)$. This is equivalent to $y = (1/3)x + 5/3$. Thus the solution is e.

(a) $y = \dfrac{2}{3}x + \dfrac{1}{3}$ (b) $y = \dfrac{2}{3}x - \dfrac{8}{3}$ (c) $y = \dfrac{1}{3}x + 4$

(d) $y = 2x + 7$ (e) $y = \dfrac{1}{3}x + \dfrac{5}{3}$

4.(6 pts.) For what value of a is the function f given by

$$f(x) = \begin{cases} \dfrac{2-x}{x^2-3x+2} & x \neq 2 \\ a & x = 2 \end{cases}$$

continuous everywhere?

Solution: We note that the function does not have $x = 1$ in its domain. Since the function is defined as a rational function of polynomials on any point other than 2, it is continuous everywhere except possibly at 2 (depending on the value of a). To ensure that the function is continuous, we need that the limit of f as x approaches from 2 from the left and right both equal $f(2) = a$. We see that

$$\lim_{x\to 2^+} f = \lim_{x\to 2^-} f = \lim_{x\to 2} \frac{2-x}{x^2-3x+2} = \lim x \to 2\frac{-1}{x-1} = -1$$

. Thus, we need $a = -1$. So the solution is b.

(a) $a = 1$

(b) $a = -1$

(c) $a = 2$

(d) $a = 0$

(e) No value of a makes f continuous everywhere

Name: ______________________________

Instructor: __________________________

5.(6 pts.) Find $f'(x)$ where

$$f(x) = \frac{\cos x}{(2x-3)^2}$$

Solution: Define $g(x) = \cos x$ and $h(x) = (2x-3)^2$, then clearly $f(x) = \frac{g(x)}{h(x)}$, so we will use the rule for differential of fraction and find that $f'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{h(x)^2}$. We remember that $g'(x) = -\sin x$ and using the chain rule we see that $h'(x) = 2(2x-3)\cdot(2x-3)' = 4(2x-3)$. So we get that

$$f'(x) = \frac{(2x-3)^2\cdot(-\sin x) - 4(2x-3)\cdot\cos x}{\left((2x-3)^2\right)^2} = \frac{-(\sin x)(2x-3)^2 - 4(\cos x)(2x-3)}{(2x-3)^4}$$

Note this could be simplified further to

$$f'(x) = \frac{-(\sin x)(2x-3) - 4(\cos x)}{(2x-3)^3}$$

(a) $\dfrac{(\sin x)(2x-3)^2 + 4(\cos x)(2x-3)}{(2x-3)^4}$

(b) $\dfrac{-(\sin x)}{4(2x-3)}$

(c) $\dfrac{-(\sin x)(2x-3)^2 - 4(\cos x)(2x-3)}{(2x-3)^4}$

(d) $\dfrac{-(\sin x)(2x-3)^2 - 4(\cos x)(2x-3)}{(2x-3)^2}$

(e) $\dfrac{4(\sin x)(2x-3) - (\sin x)(2x-3)^2}{(2x-3)^4}$

Name: ____________________________

Instructor: ________________________

6.(6 pts.) Find the equation of the line tangent to the graph of $x^4y^2 + y^3 = 2$ at the point $(1,1)$.

Solution: Recall that the tangent line through the point (x_0, y_0) is given by $y - y_0 = y' \cdot (x - x_0)$. So we need to find y' at the point $(1,1)$, for this we will use implicit differentiation. So we apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of the equation defining the graph, recall that $\frac{\mathrm{d}}{\mathrm{d}x}x^4 = 4x^3$, $\frac{\mathrm{d}}{\mathrm{d}x}y = y'$, and $\frac{\mathrm{d}}{\mathrm{d}x}y^2 = 2y \cdot y'$ (the last one follows from the chain rule, where y is viewed as a function of x). Using the product rule we get

$$4x^3 \cdot y^2 + x^4 \cdot 2y \cdot y' + 3y^2 \cdot y' = 0$$

From here we want to isolate y', and we find that

$$(2x^4y + 3y^2)y' = -4x^3y^2$$

and therefore

$$y' = \frac{-4x^3y^2}{2x^4y + 3y^2}$$

so we can evaluate this at the point in question $(x,y) = (1,1)$ and we get $y' = -\frac{4}{5}$. So returning to our original equation for the tangent line we see that $y - y_0 = y'(x - x_0)$ gives $y - 1 = -\frac{4}{5}(x-1)$ and therefore $y = -\frac{4}{5}x + \frac{9}{5}$.

(a) $y = -\frac{4}{5}x + \frac{9}{5}$

(b) $y = -\frac{1}{2}x + \frac{3}{2}$

(c) $y = \frac{4}{5}x + \frac{1}{5}$

(d) $y = -\frac{4}{5}x + \frac{4}{5}$

(e) $y = \frac{1}{2}x + \frac{1}{2}$

Name: ______________________________

Instructor: __________________________

7.(6 pts.) If

$$f(x) = \sqrt{1+\sqrt{x}}$$

what is $f'(x)$?

Solution: Define $g(x) = \sqrt{x}$ and $h(x) = 1+\sqrt{x}$, then clearly $f(x) = g(h(x))$, so by the chain rule we see that $f'(x) = g'(h(x)) \cdot h'(x)$. So let us start by finding g' and h'. $g(x) = x^{\frac{1}{2}}$ so $g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Likewise we find that $h'(x) = \frac{1}{2\sqrt{x}}$ (note that g and h only differ from each other by an added constant that disappears under differentiation). So we get that

$$f'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{2\sqrt{h(x)}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$$

(a) $\dfrac{\sqrt{x}}{4\sqrt{1+\sqrt{x}}}$ (b) $\dfrac{1}{2\sqrt{1+\sqrt{x}}}$ (c) $\dfrac{\sqrt{x}}{\sqrt{1+\sqrt{x}}}$

(d) $\dfrac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$ (e) $\dfrac{1}{\sqrt{1+\sqrt{x}}}$

8.(6 pts.) Find the linearization of the function $f(x) = \sqrt[5]{x}$ at $a = 32$ and use it to approximate the number $\sqrt[5]{34}$. Which of the following gives the resulting approximation?

SOLUTION Recall that the linearization, L, of a function at a given point is its tangent line at that point. It is given by $L(x) = f(a) + f'(a)(x-a)$. Note that

$$\frac{d}{dx}\, x^{1/5} = \frac{1}{5}x^{-4/5}$$

Hence $f'(a) = f'(32) = \frac{1}{5}\frac{1}{(2^5)^{4/5}} = \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80}$. Hence $L(x) = 2 + \frac{1}{80}(x-32)$.

We use $L(x)$ to approximate $\sqrt[5]{34}$. By computing

$$\sqrt[5]{34} \approx L(34) = 2 + \frac{1}{80}(34-32) = 2 + \frac{1}{40} = \frac{2\cdot 40+1}{40} = \frac{81}{40}.$$

(a) $\dfrac{21}{20}$ (b) $\dfrac{79}{40}$ (c) $\dfrac{19}{20}$ (d) $\dfrac{81}{40}$ (e) 2

Name: ____________________________

Instructor: _________________________

9.(6 pts.) Two cyclists are approaching a town, one cycling due east at 10 miles per hour and the other cycling due south at 15 miles per hour. How fast is the distance between the bicycles decreasing when the eastbound cyclist is 40 miles from the town and the southbound cyclist is 30 miles from the town?

SOLUTION

Let us denote the distance between the cyclist due east and the town by $X(t)$ and $Y(t)$ for the one due south. We have $\frac{dX}{dt} = -10$ *mph* and $\frac{dY}{dt} = -15$ *mph*. Notice that their trajectories relatives to the town form a right triangle. Hence if we denote the distance between the bicycles by $D(t)$ we have by the Pythagorean Theorem that $D(t)^2 = X(t)^2 + Y(t)^2$. Again, by the Pythagorean Theorem $D\big|_{X=40,Y=30} = 50$. By taking derivatives in both sides we obtain

$$\frac{d}{dt}(D(t)^2) = \frac{d}{dt}(X(t)^2 + Y(t)^2) \Longrightarrow \;\; 2D(t)\cdot\frac{dD}{dt} = 2X(t)\cdot\frac{dX}{dt} + 2Y(t)\cdot\frac{dY}{dt}$$

$$\Longrightarrow \;\; \frac{dD}{dt} = \frac{X(t)}{D(t)}\cdot\frac{dX}{dt} + \frac{Y(t)}{D(t)}\cdot\frac{dY}{dt} \Longrightarrow \frac{dD}{dt}\Big|_{X=40,Y=30} = \frac{40}{50}\cdot(-10) + \frac{30}{50}\cdot(-15) = -17$$

We conclude that their distance D, when $X = 40$ and $Y = 30$, is decreasing at a rate of 17 *mph*.

(a) 17 m. p.h. (b) 32 m.p.h. (c) 15 m.p.h.

(d) 20 m.p.h. (e) 12 m.p.h.

10.(6 pts.) Among all positive x values (for $0 < x < \infty$), find the value of x which minimizes

$$y = f(x) = x + \frac{3}{x}.$$

Solution: We compute $f'(x) = 1 - \frac{3}{x^2}$. Thus, $f'(x) = 0$ implies that $1 - \frac{3}{x^2} = 0$, so $x^2 = 3$, and since $x > 0$, $x = \sqrt{3}$. Checking that $f'(x) < 0$ for $0 < x < \sqrt{3}$ and $f'(x) > 0$ for $x > \sqrt{3}$ ensures that $x = \sqrt{3}$ gives the minimum value of $f(x)$.

(a) $x = 1.5$ (b) $x = \sqrt{2}$

(c) $x = 1$ (d) $x = \sqrt{3}$

(e) No such value of x exists.

Name: ______________________________

Instructor: __________________________

11.(6 pts.) Let

$$f(x) = \frac{|x|}{x^2+1}.$$

Which of the following statements is true about $f(x)$?

Solution: First, for $x > 0$, note that $f(x) = \frac{x}{x^2+1}$ and for $x < 0$, note that $f(x) = \frac{-x}{x^2+1}$. Using the quotient rule, we see that $f'(x) = \frac{1-x^2}{(x^2+1)^2}$ for $x > 0$ and $f'(x) = \frac{x^2-1}{(x^2+1)^2}$ for $x < 0$. Since the denominator is never 0, it follows that the critical points are at $x = 0$ (due to the absolute value), and where the numerator is zero, which gives $x = \pm 1$. One checks that $f'(x) > 0$ for $x < -1$, $f'(x) < 0$ for $-1 < x < 0$, $f'(x) > 0$ for $0 < x < 1$, and $f'(x) < 0$ for $x > 1$. It now follows that we have local maxima at $x = \pm 1$ and a local minimum at $x = 0$.

(a) $f(x)$ has a local maximum at $x = 1$, and no local minima.

(b) There is a local minimum at $x = -1$ and a local maximum at $x = 1$

(c) $f(x)$ has no critical points besides $x = -1$ and $x = 1$.

(d) $f(x)$ has no critical points.

(e) There are local maxima at $x = \pm 1$, and a local minimum at $x = 0$

12.(6 pts.) How many inflection points does the graph of $f(x) = \sin(x) - \cos(x)$ have on the interval $[0, \pi]$.

Solution: We see that $f'(x) = \cos(x) + \sin(x)$, and $f''(x) = -\sin(x) + \cos(x)$. Thus, $f''(x) = 0$ implies $\sin(x) = \cos(x)$, or $\tan(x) = 1$. For $x \in [0, \pi]$, $\tan(x) = 1$ implies that $x = \frac{\pi}{4}$. For $0 < x < \frac{\pi}{4}$, $f''(x) > 0$, and for $\frac{\pi}{4} < x < \pi$, $f''(x) < 0$. Thus, there is only one inflection point, at $\frac{\pi}{4}$.

(a) 2 (b) 1 (c) 3 (d) 4 (e) 0

Name: ______________________________

Instructor: ___________________________

13.(6 pts.) Find the limit

$$\lim_{x\to 0^+} \frac{\sin(x^2)}{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sin(x^2)}{x} &= \lim_{x\to 0^+} \frac{\sin(x^2)}{x^2} \cdot x \\
&= \lim_{x\to 0^+} \frac{\sin(x^2)}{x^2} \lim_{x\to 0^+} x \\
&= \lim_{x^2\to 0} \frac{\sin(x^2)}{x^2} \lim_{x\to 0^+} x \\
&= 1 \cdot 0 \\
&= 0
\end{aligned}$$

(a) 0

(b) $-\infty$

(c) 1

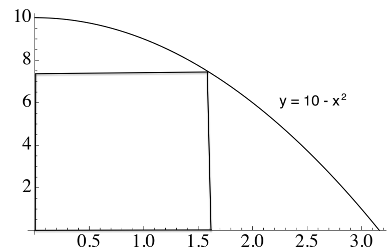
(d) $+\infty$

(e) Does not exist and is not equal to $\pm\infty$

Name: ____________________________

Instructor: ________________________

14.(6 pts.) Consider the picture shown below. You wish to fit a rectangle of maximal area into the region bounded by the curve $y = 10 - x^2$ and the lines $y = 0$ and $x = 0$, such that one corner of the rectangle is positioned at the point $(0, 0)$ and the opposite corner touches the graph of the curve as shown. What are the dimensions of such a rectangle of maximal area?



Solution:

The area of the rectangle is given by

$$A = xy = x(10 - x^2) = 10x - x^3,$$

where $0 \le x \le \sqrt{10}$.

To maximizes A, we take the derivative of A with respect to x:

$$A'(x) = 10 - 3x^2.$$

Within the dommain $[0, \sqrt{10}]$, we have only one critical point: $\sqrt{\frac{10}{3}}$.

Since $A'(x) > 0$ for $0 \le x < \sqrt{\frac{10}{3}}$ and $A'(x) < 0$ for $\sqrt{\frac{10}{3}} < x \le \sqrt{10}$, it follows that the maximum area occurs at $x = \sqrt{\frac{10}{3}}$ and $y = 10 - \frac{10}{3} = \frac{20}{3}$.

(a) $\dfrac{\sqrt{10}}{2} \times \dfrac{30}{4}$

(b) 2×6

(c) $\sqrt{\dfrac{10}{3}} \times \dfrac{20}{3}$

(d) $\sqrt{3} \times 7$

(e) $\dfrac{\sqrt{10}}{2} \times \dfrac{\sqrt{10}}{2}$

Name: ____________________________

Instructor: _________________________

15.(6 pts.) The equation $\tan(x) - \sin(2x) - \dfrac{1}{2} = 0$ has one solution between 0 and 1.

Find the result of one iteration of Newton's Method applied to this equation with 0 as the starting point. (i.e. find x_2 using Newton's method applied to the equation with $x_1 = 0$).

Solution:

Let $f(x) = \tan(x) - \sin(2x) - \frac{1}{2}$. It follows that $f'(x) = \sec^2(x) - 2\cos(2x)$, and then

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0 - \frac{f(0)}{f'(0)} \\ &= -\frac{\tan(0) - \sin(0) - \frac{1}{2}}{\sec(0) - 2\cos(0)} \\ &= -\frac{-\frac{1}{2}}{1-2} \\ &= -\frac{1}{2}. \end{aligned}$$

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{3}$ (c) -1 (d) $\dfrac{1}{3}$ (e) $-\dfrac{1}{2}$

Name: ____________________________

Instructor: ________________________

16.(6 pts.) Consider the following rational function:

$$f(x) = \frac{(x-1)(x^2+1)}{x^2+x-2}.$$

Which of the statements shown below is true?

Solution: $f(x) = \frac{(x-1)(x^2+1)}{x^2+x-2} = \frac{(x-1)(x^2+1)}{(x+2)(x-1)}$. We compute

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x^2+1)}{(x+2)(x-1)} = \lim_{x\to 1} \frac{x^2+1}{x+2} = \frac{2}{3}.$$

Hence $x = 1$ is a removable discontinuity of $f(x)$. So $x = 1$ is not an asymptote of $f(x)$.

$f(x) = \frac{(x-1)(x^2+1)}{x^2+x-2} = \frac{x^3-x^2+x-1}{x^2+x-2}$. We compute

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3-x^2+x-1}{x^2+x-2} = \lim_{x\to\infty} \frac{x-1+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x}} = \infty.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3-x^2+x-1}{x^2+x-2} = \lim_{x\to-\infty} \frac{x-1+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x}} = -\infty.$$

Therefore $f(x)$ has no horizontal asymptote.

Finally we perform long division

$$\begin{array}{r|l}
 & x-2 \\
\hline
x^2+x-2 & x^3 - x^2 + x - 1 \\
 & -x^3 - x^2 + 2x \\
\hline
 & -2x^2 + 3x - 1 \\
 & 2x^2 + 2x - 4 \\
\hline
 & 5x - 5
\end{array}$$

and see that $f(x) = \frac{(x-1)(x^2+1)}{x^2+x-2} = x - 2 + \frac{5x-5}{x^2+x-2}$. Therefore $f(x)$ has a slant asymptote $y = x - 2$.

(a) $x = 1$ is a vertical asymptote of f.

(b) $y = x - 2$ is a slant asymptote of f.

(c) $y = 1$ is a horizontal asymptote of f

(d) $y = 2$ is a horizontal asymptote of f

(e) $y = x + 2$ is a slant asymptote of f.

Name: ___________________________

Instructor: _______________________

17.(6 pts.) Consider the definite integral

$$\int_1^3 x^2 + 1 \, dx.$$

Which of the following Riemann sums gives the **right end point approximation** to the above integral, using four approximating rectangles?

Solution: We write the Riemann sum using right end-point approximation with four intervals:

$$\sum_{i=1}^{n} f(a + \frac{b-a}{n} i)(\frac{b-a}{n})$$

(Note $f(x) = x^2 + 1, b = 3, a = 1, n = 4$)

$$= \sum_{i=1}^{4} ((1 + \frac{(3-1)i}{4})^2 + 1)(\frac{3-1}{4})$$

$$= \frac{1}{2}(((\frac{3}{2})^2 + 1) + (2^2 + 1) + ((\frac{5}{2})^2 + 1) + (3^2 + 1))$$

$$= \frac{1}{2}(\frac{13}{4} + 5 + \frac{29}{4} + 10)$$

(a) $\frac{1}{2}\left(2 + \frac{13}{4} + 5 + \frac{29}{4}\right)$

(b) $\frac{1}{2}\left(\frac{9}{4} + 4 + \frac{25}{4} + 9\right)$

(c) $\frac{1}{2}\left(\frac{13}{4} + 5 + \frac{29}{4} + 10\right)$

(d) $\frac{1}{2}\left(1 + \frac{9}{4} + 4 + \frac{25}{4}\right)$

(e) $\frac{1}{2}\left(\frac{25}{4} + 9 + \frac{49}{4} + 16\right)$

Name: ____________________________

Instructor: ________________________

18.(6 pts.) Let

$$F(x) = \int_1^{x^3} \sqrt{1+\sin^2(u)}\, du.$$

Which of the following gives $F'(x)$?

Solution: Let $f(u) = \sqrt{1+\sin^2(u)}$ and $g(x) = x^3$. Then we see that
$F'(x) = \frac{d}{dx}\int_1^{x^3} \sqrt{1+\sin^2(u)}\, du. = \frac{d}{dx}\int_1^{g(x)} f(u)\, du. = f(g(x))g'(x)$ via the chain rule.
Therefore $F'(x) = f(g(x))g'(x) = 3x^2\sqrt{1+\sin^2(x^3)}$ which is answer choice (e).

(a) $\dfrac{\sin(x^3)\cos(x^3)}{\sqrt{1+\sin^2(x^3)}}$ (b) $\dfrac{\sin(x)\cos(x)}{\sqrt{1+\sin^2(x)}}$ (c) $3x^2\sqrt{1+\sin^2(x)}$

(d) $\dfrac{3x^2\sin(x^3)\cos(x^3)}{\sqrt{1+\sin^2(x^3)}}$ (e) $3x^2\sqrt{1+\sin^2(x^3)}$

Name: ______________________________

Instructor: __________________________

19.(6 pts.) The function $F(x)$ has the following properties:

$$F'(x) = \sin(4x) \quad \text{and} \quad F(0) = 3/4.$$

Find $F\left(\frac{\pi}{8}\right)$.

Solution: $F(x) = \int F'(x)dx = \int \sin(4x)dx = \frac{-\cos(4x)}{4} + C$. Additionally, we know the initial condition so $F(0) = 3/4 = \frac{-\cos(4*0)}{4} + C = \frac{-1}{4} + C$ and so $C = 1$.

Therefore $F(x) = \frac{-\cos(4x)}{4} + 1$.

We are asked to find $F(\frac{\pi}{8}) = \frac{-\cos(\frac{4\pi}{8})}{4} + 1 = F(x) = \frac{0}{4} + 1 = 1$. This is answer (c).

(a) $\frac{7}{4}$ (b) 0 (c) 1 (d) $\frac{3}{4}$ (e) 2

20.(6 pts.) Evaluate the following definite integral

$$\int_0^1 \frac{x^2}{\sqrt{2x^3+1}}\,dx$$

Solution: We use the method of substitution. Let $u(= u(x)) = 2x^3 + 1$, we get

$\frac{du}{dx} = 6x^2$, so we replace $x^2 dx$ by $\frac{du}{6}$ and we replace $\sqrt{2x^3+1}$ by $\sqrt{u}$.

We get $u(0) = 2(0)^3 + 1 = 1$ and $u(1) = 2(1)^3 + 1 = 3$.

$$\int_0^1 \frac{x^2}{\sqrt{2x^3+1}}\,dx = \int_1^3 \frac{1}{6\sqrt{u}}\,du = \frac{2\sqrt{u}}{6}\bigg|_1^3 = \frac{\sqrt{3}}{3} - \frac{\sqrt{1}}{3} = \frac{\sqrt{3}-1}{3}$$

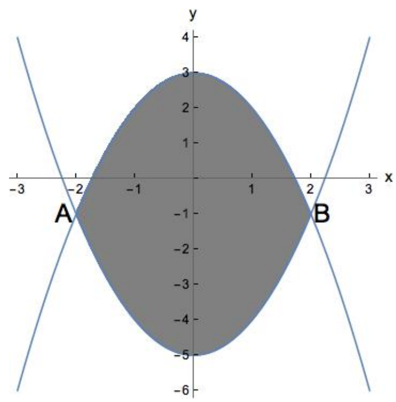
(a) $\frac{\sqrt{3}-1}{3}$ (b) $2\sqrt{3}-1$ (c) $\frac{1}{3}$ (d) $\frac{\sqrt{2}-1}{3}$ (e) $\frac{1}{6}$

Name: ____________________________

Instructor: ________________________

21.(6 pts.) Find the area between the curves $y = 3 - x^2$ and $y = x^2 - 5$.

Solution: Its a good idea to draw a rough sketch of the graph here, noting that $y = x^2 - 5$ is just a shift of the graph of $y = x^2$ downwards by 5 units and the graph of $y = 3 - x^2$ is a shift of the graph of $y = -x^2$ (= graph of $y = x^2$ upsidedown) upwards by 3 units.



Even if you don't draw a rough sketch of the graph, the first step in solving this problem is to solve for where the curves cross, that is where :

$$3 - x^2 = x^2 - 5 \text{ that is when } 8 - 2x^2 = 0 \text{ when } x = \pm 2.$$

If we let $f(x) = 3 - x^2$ and $g(x) = x^2 - 5$, both functions are continuous and hence $f(x) - g(x) = 8 - 2x^2$ is continuous. Thus we can determine if it positive or negative on the interval $(-2, 2)$ by checking at a single point. Since $f(0) - g(0) = 8 > 0$, thus $f(x) - g(x) > 0$ on the interval $(-2, 2)$ and the area between the curves is given by

$$\int_{-2}^{2} f(x) - g(x)\, dx = \int_{-2}^{2} 8 - 2x^2\, dx = \left(8x - \frac{2x^3}{3}\right)\Bigg|_{-2}^{2}$$

$$= \left(16 - \frac{16}{3}\right) - \left(-16 - \frac{(-16)}{3}\right) = \frac{64}{3}$$

(a) $\frac{64}{3}$ (b) 0 (c) 16 (d) 32 (e) $\frac{32}{3}$

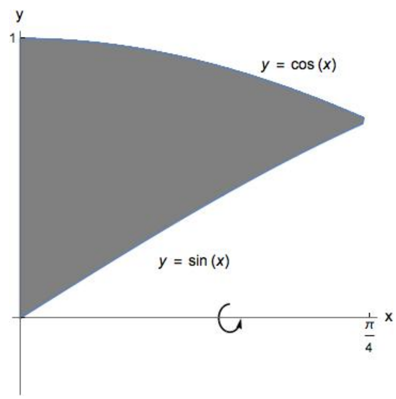
Name: ______________________________

Instructor: __________________________

22.(6 pts.) A region in the xy plane is bounded by the curves $y = \cos x$, $y = \sin x$, $x = 0$ and $x = \frac{\pi}{4}$. Which integral below gives the volume of the solid obtained by rotating the given region about the x-axis?

Solution:

Note: First off note that since we are rotating a region bounded by functions of x about the x-axis, the method of discs or washers will be used.
A good sketch of the region can help here and since the curves are basic this is possible to draw without too much difficulty.



[One needs to check whether the curves meet in the interval $(0, \dfrac{\pi}{4})$ and which one is larger. You may have already done this from your sketch, but if in doubt, you can use the unit circle to figure everything out. We know from the unit circle that $\sin(x) = \cos(x)$ at only point in the interval $[0, \frac{\pi}{4}]$, that is when $x = \frac{\pi}{4}$, hence we need only determine which one is larger on the interval. We can figure this out from the unit circle or alternatively we can use the fact that the function $\cos(x) - \sin(x)$ is continuous on the interval $(-\frac{\pi}{4}, \frac{\pi}{4})$ and has no zeros on that interval, hence since $\cos(0) - \sin(0) > 0$, it must be positive on the entire interval.]

We use the special case of the formula for volume

$$V = \int_a^b A(x)\, dx$$

where $A(x)$ denotes the area of the cross section (perpendicular to the x-axis) of the solid at x. In this case the cross section has a washer shape with area

$$\pi((\text{outer radius})^2 - (\text{inner radius})^2) = \pi(\cos^2(x) - \sin^2(x)).$$

Thus the volume of the solid we get by rotating the above region around the x axis is given by

$$V = \int_0^{\frac{\pi}{4}} \pi[\cos^2 x - \sin^2 x]\, dx.$$

Name: ______________________________

Instructor: __________________________

(a) $\displaystyle\int_0^{\frac{\pi}{4}} 2\pi x[\cos^2 x - \sin^2 x]\ dx$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} 2\pi x[\cos x - \sin x]\ dx$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} \pi[\sin^2 x - \cos^2 x]\ dx$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \pi[\cos^2 x - \sin^2 x]\ dx$

(e) $\displaystyle\int_0^{\frac{\pi}{4}} \pi x[\sin^2 x - \cos^2 x]\ dx$

Name: ______________________________

Instructor: __________________________

23.(6 pts.) A force of 10 lbs is required to hold a spring stretched 2 feet beyond its natural length. How much work is done in stretching the spring from its natural length to one foot beyond its natural length?

Solution: According to Hooke's law, the force required to hold a spring x feet beyond its natural length is $F(x) = kx$ lbs for some constant k. In this case, we know the force required when $x = 10$ and we can use that to find the spring constant:

$$F(2) = 10 = k2, \qquad \text{therefore} \qquad k = 5.$$

The work done in stretching the spring from its natural length to one foot beyond its natural length is

$$W = \int_0^1 F(x)\,dx = \int_0^1 5x\,dx = \frac{5x^2}{2}\Big|_0^1 = \frac{5}{2} = 2.5 \text{ ft lb}.$$

(a) 5 ft lb (b) 10 ft lb (c) 2.5 ft lb (d) 1 ft lb (e) 1.5 ft lb

Name: ______________________________

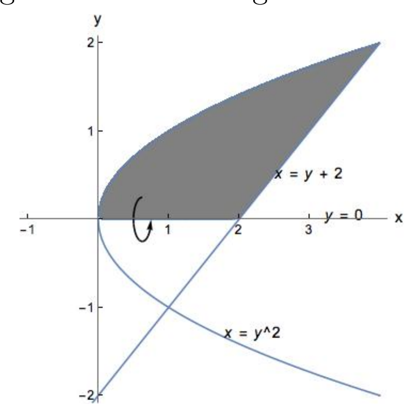
Instructor: __________________________

24.(6 pts.) Which integral below gives the volume of the solid generated by rotating the region enclosed by the curves $x = 2 + y$, $x = y^2$ and $y = 0$ about the $x-$axis.

Solution:

Note: First off note that since we are rotating functions of y around the x-axis, the shell method is the appropriate method to use to find the volume.

It is advisable to draw a rough sketch of the region:



We will use the formula for the shell method when rotating a region between two functions of y on $[a, b]$ around the x-axis where $f(y) - g(y) \geq 0$ on $[a, b]$:

$$V = \int_a^b 2\pi y[f(y) - g(y)]\, dy.$$

In order to determine the limits of integration, we need to find the positive y where the above two functions meet, that is the value of y for which $2 + y = y^2$ or $y^2 - y - 2 = 0$. This happens when $(y - 2)(y + 1) = 0$, that is $y = 2$ or $y = -1$. Thus our limits of integration are $y = 0$ and $y = 2$.

At $y = 1$, we have $2 + y = 3 > 1 = y^2$, therefore by continuity $2 + y - y^2 \geq 0$ for $0 \leq y \leq 2$ and thus our volume is given by:

$$V = \int_0^2 2\pi y[2 + y - y^2]\, dy.$$

Less Long winded Solution :)

Note that the curves $x = 2 + y$ and $x = y^2$ intersects at two points:

$$(-1, -1); \quad (4, 2).$$

We are interested in the region enclosed by the curves $x = 2 + y$, $x = y^2$ and $y = 0$ that lies in the first quadrant, so $(4, 2)$ is the point we consider.

The cylindrical shell method yields:

$$V = \int_0^2 2\pi y[(2 + y) - y^2]\, dy = 2\pi \int_0^2 y[(2 + y) - y^2]\, dy.$$

Name: ______________________________

Instructor: __________________________

Incidentally, the disk/washer method yields:

$$V = \int_0^2 \pi x \, dx + \int_2^4 \pi[x - (x-2)^2] \, dx.$$

(a) $\pi \int_0^2 y[(2+y) - y^2] \, dy$

(b) $2\pi \int_0^2 [(2+y)^2 - y^4] \, dy$

(c) $2\pi \int_0^2 y[(2+y)^2 - y^4] \, dy$

(d) $2\pi \int_0^2 y[2 + y - y^2] \, dy$

(e) $\pi \int_0^2 y[(2+y)^2 - y^4] \, dy$

Name: ______________________________

Instructor: __________________________

25.(6 pts.) Find the average value of $f(x) = 2\sin x \cos x$ over the interval $[0, \frac{\pi}{4}]$.

Solution : Note that $f(x) = \sin(2x)$, so

$$\int_0^{\pi/4} \sin(2x)dx = -\frac{cos(2x)}{2}\Big|_0^{\pi/4} = -\frac{\cos(\pi/2)}{2} - -\frac{\cos(0)}{2} = \frac{1}{2}.$$

Since the average value of $f(x)$ on $[0, \frac{\pi}{4}]$ is

$$\frac{1}{\pi/4 - 0}\int_0^{\pi/4} \sin(2x)dx,$$

the average value is

$$\frac{1/2}{\pi/4} = \frac{2}{\pi}.$$

(a) $\frac{\pi}{2}$ (b) $\frac{4}{\pi}$ (c) π (d) $\frac{2}{\pi}$ (e) $\frac{\pi}{4}$

Name: ____________________

Math 10550, Final Exam: Solutions
December 18, 2016

Instructor: ANSWERS

- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.
- No calculators are to be used.
- The exam lasts for two hours.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 22 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(•)	(e)	15.	(a)	(b)	(c)	(d)	(•)
2.	(a)	(b)	(c)	(d)	(•)	16.	(a)	(•)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(•)	17.	(a)	(b)	(•)	(d)	(e)
4.	(a)	(•)	(c)	(d)	(e)	18.	(a)	(b)	(c)	(d)	(•)
5.	(a)	(b)	(•)	(d)	(e)	19.	(a)	(b)	(•)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)	20.	(•)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(•)	(e)	21.	(•)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(•)	(e)	22.	(a)	(b)	(c)	(•)	(e)
9.	(•)	(b)	(c)	(d)	(e)	23.	(a)	(b)	(•)	(d)	(e)
10.	(a)	(b)	(c)	(•)	(e)	24.	(a)	(b)	(c)	(•)	(e)
11.	(a)	(b)	(c)	(d)	(•)	25.	(a)	(b)	(c)	(•)	(e)
12.	(a)	(•)	(c)	(d)	(e)						
13.	(•)	(b)	(c)	(d)	(e)						
14.	(a)	(b)	(•)	(d)	(e)						